extension, or whether the free parts of those fields which play such a role need, in addition, to be nonlinear in some specific way, or whether only general relativistic coupling will work. Equation (1) would still hold for such extended theories, being a purely kinematical property of homogeneous space-time, but the usual local commutativity properties would be lost; that is, the commutator of a field component with its time-derivative would, even at equal times, depend on other fields. Alternately, the conjugate momentum would be a function of such other fields as well as of the time derivative of the corresponding component. The assumption that this is not the case appears to be implicitly required in the proofs of the pessimistic theorems. The commutation relations are closely related to the creation and annihilation of particles; if they now depend on the other fields present, it might happen that the contributions, as the thresholds of higher and higher creation processes are passed with increasing energies, are damped thereby. These contributions from new creation processes seem to be the cause (or another expression) of the divergences in field theory, in which cases the new couplings might yield convergence. Such couplings arise, for example, in quantum hydrodynamics.[19] Saturation might be expected to occur with some of these couplings; that is, the presence of many quanta, or high energies, may damp further creation.[20]

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I am most thankful to Professor O. Klein for many valuable discussions of this work; his contributions to all its phases have been extremely illuminating.

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[19] A. Thellung, Helv. Phys. Acta **29**, 103 (1956). In this case the kinetic energy has the form $\rho v^2/2$, both ρ and v being fields. This form may be very different, however, from one in which several variables ρ_{ij} multiply v_iv_j.

[20] Landau[17] has also remarked that at high energies field theory might go over into a quantum hydrodynamical scheme.

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Negative Mass in General Relativity

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1.

Although some of the arguments relating to negative mass are fairly elementary and well known, it will nevertheless be of advantage to rediscuss the meaning of this term. In the first instance, without fully specifying a theory, we can distinguish between three kinds of mass according to the measurement by which it is defined: inertial, passive gravitational, and active gravitational mass. Inertial mass is the quantity that enters (and is defined by) Newton's second law*; passive gravitational mass is the mass on which the gravitational fields acts, that is it is defined by $\mathbf{F}=-m$ grad$V$; active gravitational mass is the mass that is the source of gravitational fields and is hence the mass that enters Poisson's equation and Gauss' law.

In Newtonian physics the law of action and reaction implies the equality of active and passive gravitational masses, but the equality of inertial mass with these other two is a separate empirical fact. The sign of both these masses can take either value and it is an additional empirical result that it is always positive. Four cases accordingly arise, if this empirical fact is left out of account.

(i) All mass is positive; this is familiar.

(ii) Inertial mass negative, gravitational mass is positive. A body consisting of matter of this kind will respond perversely to all forces whether gravitational or of other kinds, but will produce gravitational forces just as a usual body does.

(iii) Inertial mass positive, gravitational masses negative. In this case we would have normal behavior relating to all nongravitational forces, but gravitational behavior involving masses of this type and of type (i) would be governed by a negative Coulomb law; i.e., like masses would attract and unlike masses would repel.

(iv) All mass is negative. This would be a combination of (ii) and (iii). Matter of this kind responds perversely to nongravitational forces, responds like ordinary matter to gravitational forces, but produces repulsive gravitational fields.

In general relativity the situation is quite different. The principle of equivalence is not a separate fact but is basic to the theory. Accordingly the ratio of inertial and passive gravitational masses is the same for all bodies. The relation between active and passive gravitational masses is not fixed by anything like Newton's third law as this would require integrals over

* A mass-independent force (say, of electromagnetic nature) has to be used here, for obvious reasons.

extended regions of space-time which do not possess the required tensorial character. A good deal of work has been done[1,2] and tends to indicate that the relation is rather complicated.

As long as relativity is considered purely as a theory of gravitation, the inertial and passive gravitational masses do not in fact appear. Active gravitational mass occurs for the first time as a constant of integration in Schwarzschild's solution. If this constant is taken to be positive, then test particles will, in the first approximation, describe the Newtonian orbits corresponding to an attractive body. If, however, the constant is taken to be negative then, in the first approximation, test particles will describe the orbits corresponding to the Newtonian case with repulsion. Note that in the first case *all* bodies will be attracted, in the second *all* bodies will be repelled.

If we now leave the one-body problem and consider the two-body case, then a remarkable situation arises. Imagine a body of positive mass and a body of negative mass separated by empty space. Then, to use the language of the Newtonian approximation, the positive body will attract the negative one (since all bodies are attracted by it), while the negative body will repel the positive body (since all bodies are repelled by it). If the motion is confined to the line of centers, then one would expect the pair to move off with uniform acceleration. This rather surprising result clearly requires confirmation by the complete construction of the model in general relativity.

2.

Uniformly accelerated systems in general relativity (and in special relativity) are well known,[3] but it may be worth while repeating here briefly the main properties of such systems. The Newtonian concept of uniform acceleration may be generalized to special relativity in a number of ways, but one of these is of outstanding importance in retaining the stationary property. The system in this case is described by the equation $\zeta^2-\tau^2=\text{const}$. The orbits of all particles in the $\tau-\zeta$ plane form a system of rectangular hyperbolas with fixed asymptotes. These particles all have an acceleration which is uniform in the sense that the motion of each particle viewed from that particle is constant in time. However, although the acceleration of every point is uniform, the acceleration of different points is not the same. Roughly speaking, the nearer the trajectory of the particle passes to the origin the larger its acceleration. The remarkable feature of this system is its stationary character. If any particle carried an observer measuring the distance of any other particle partaking of the motion, then he would find this distance to be constant in time, although to a fixed observer he and the other particle would have different accelerations. The usual Minkowski metric may be transformed to the uniformly accelerated frame by the transformation $\tau=z\sinh t$, $\zeta=z\cosh t$, $\xi=x$, $\eta=y$, resulting in the metric $ds^2=z^2dt^2-dx^2-dy^2-dz^2$. It is clear that this metric does not cover the whole of space, but is bordered by portions of the asymptotes which act as horizons and accordingly the metric only covers one-quarter of space-time (Fig. 1).

We now return to the task of constructing a model in general relativity of two bodies whose masses have opposite sign. Accordingly we use a uniformly accelerated frame, and then immerse in it two finite bodies with opposite sign of mass. In the uniformly accelerated frame the system will be axially symmetric and so we can use[4] the metric of Weyl and Levi-Civita. In this metric we have, in empty space,

$$ds^2=e^{2\varphi}d\bar{t}^2-e^{-2\varphi}[e^{2\sigma}(d\bar{r}^2+d\bar{z}^2)+\bar{r}^2d\theta^2] \tag{1}$$

where $\varphi=\varphi(\bar{r},\bar{z})$, $\sigma=\sigma(\bar{r},\bar{z})$ satisfy

$$\left[\frac{\partial^2}{\partial\bar{r}^2}+\frac{1}{\bar{r}}\frac{\partial}{\partial\bar{r}}+\frac{\partial^2}{\partial\bar{z}^2}\right]\varphi=0. \tag{2}$$

[The operator occurring in (2) will be denoted by ∇^2.]

$$\frac{\partial\sigma}{\partial\bar{r}}=\bar{r}\left[\left(\frac{\partial\varphi}{\partial\bar{r}}\right)^2-\left(\frac{\partial\varphi}{\partial\bar{z}}\right)^2\right] \tag{3}$$

$$\frac{\partial\sigma}{\partial\bar{z}}=2\bar{r}\frac{\partial\varphi}{\partial\bar{r}}\frac{\partial\varphi}{\partial\bar{z}}. \tag{4}$$

It is well known that there is a consistency condition for this metric.[4] For our purposes this condition will be given in a slightly different form from the usual one.

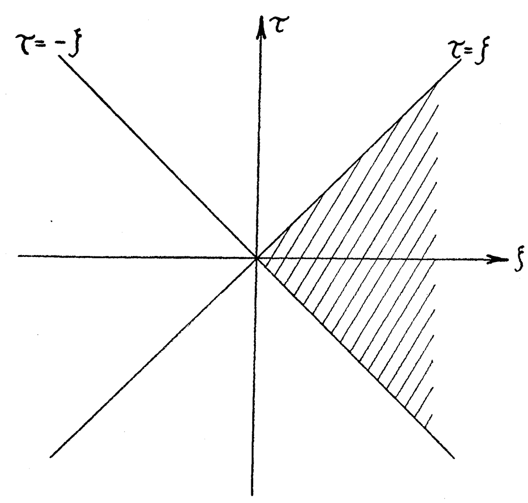

FIG. 1. The shaded portion on the right is mapped on the half-space $z\geqslant 0$ and also on the entire $(\bar{t},\bar{r},\bar{z},\theta)$ space.

[1] J. L. Synge, Proc. Edinburgh Math. Soc. (2) **7**, 93 (1937).
[2] E. T. Whittaker, Proc. Roy. Soc. (London) **A149**, 384 (1935).
[3] L. Marder, Proc. Cambridge Phil. Soc. **53**, 194 (1957).

[4] P. G. Bergmann, *Introduction to the Theory of Relativity* (Prentice Hall, Inc., New York, 1946), p. 208.

For any closed circuit C situated entirely in empty space (though possibly enclosing a nonempty region) we have from (3) and (4)

$$0=\oint_C\left[\frac{\partial\sigma}{\partial\bar r}d\bar r+\frac{\partial\sigma}{\partial\bar z}d\bar z\right]=\oint_C\left\{\bar r\left[\left(\frac{\partial\varphi}{\partial\bar r}\right)^2-\left(\frac{\partial\varphi}{\partial\bar z}\right)^2\right]d\bar r+2\bar r\frac{\partial\varphi}{\partial\bar r}\frac{\partial\varphi}{\partial\bar z}d\bar z\right\}=2\int\frac{\partial\varphi}{\partial\bar z}\nabla^2\varphi\,\bar r d\bar r d\bar z \quad (5)$$

by the divergence theorem, the integral being taken through the part of the meridian plane enclosed by C. It has been assumed here that through the whole of this region the metric has the form (1) though it need not satisfy Eqs. (2), (3), and (4).

The metric (1) is not in fact the most general metric for a nonempty static axially symmetric region but it is sufficiently general for the model we wish to construct. It is easy to show that

$$-\kappa\rho=-\kappa T_0{}^0=e^{2(\varphi-\sigma)}\left[-2\nabla^2\varphi+\nabla^2\sigma-\frac{1}{\bar r}\frac{\partial\sigma}{\partial\bar r}+\left(\frac{\partial\varphi}{\partial\bar r}\right)^2+\left(\frac{\partial\varphi}{\partial\bar z}\right)^2\right] \quad (6)$$

$$-\kappa p_{11}=-\kappa T_1{}^1=\kappa T_2{}^2=\kappa p_{22}=e^{2(\varphi-\sigma)}\left[\frac{1}{\bar r}\frac{\partial\sigma}{\partial\bar r}-\left(\frac{\partial\varphi}{\partial\bar r}\right)^2+\left(\frac{\partial\varphi}{\partial\bar z}\right)^2\right] \quad (7)$$

$$-\kappa p_{33}=-\kappa T_3{}^3=e^{2(\varphi-\sigma)}\left[\nabla^2\sigma-\frac{1}{\bar r}\frac{\partial\sigma}{\partial\bar r}+\left(\frac{\partial\varphi}{\partial\bar r}\right)^2+\left(\frac{\partial\varphi}{\partial\bar z}\right)^2\right] \quad (8)$$

$$-\kappa T_{12}=2\frac{\partial\varphi}{\partial\bar r}\frac{\partial\varphi}{\partial\bar z}-\frac{1}{\bar r}\frac{\partial\sigma}{\partial\bar z}. \quad (9)$$

The type of specialization involved in the retention of metric (1) is clearly shown in (7). The condition that T_{12} is finite on the axis implies that $\sigma=\text{const}$ on $\bar r=0$. Without loss of generality we may take $\sigma=0$ on $\bar r=0$. If we suppose that there is no matter on $\bar r=0$, then this condition also follows from (5).

We can now construct a Newtonian analog of our system in which $\bar r$, $\bar z$, θ are cylindrical polar coordinates and in which φ is the gravitational potential (in gravitational units). In empty space φ satisfies Laplace's equation (2). Also, as long as φ is small, (3) and (4) imply that σ is small of the second order. Equation (6) is then, to the first order, identical with Poisson's equation, while Eqs. (7), (8), and (9) imply that the stresses are small compared with the density.

The significance of the consistency condition (5) is now clear: it is that the gravitational force on any body surrounded by empty space must have a vanishing $\bar z$ component. Since the other components vanish by symmetry, (5) is simply the Newtonian equilibrium condition. We can now regard φ as the *exact* Newtonian potential of a Newtonian analog system though the density of the Newtonian system will not be exactly the same as the density of the relativistic system.

Since Laplace's equation is linear, we can of course superpose solutions. In particular if there are two bodies [i.e., two separate regions in which (2) does not hold], then the two corresponding solutions may be superposed, subject to condition (5) holding for each body separately. A few theorems of Newtonian gravitation may now usefully be quoted.

(i) If $\varphi\to 0$ at ∞ and $\nabla^2\varphi=0$ except in a single finite closed region, then condition (5) is satisfied.

(ii) If $\varphi\to 0$ at ∞ and $\nabla^2\varphi=0$ except in a finite closed region which lies entirely in the region $\bar z<a$ and in which $\nabla^2\varphi\geqslant 0$, then $\partial\varphi/\partial\bar z\geqslant 0$ for all $\bar z\geqslant a$.

(iii) If $\varphi\to 0$ at ∞ and $\nabla^2\varphi=0$ except in two finite regions one of which lies entirely in $\bar z<a$ and the other in $\bar z>a$, and in each of which $\nabla^2\varphi$ is of one sign, then condition (5) cannot be satisfied. This important result follows from (i) and (ii). If φ is split into two parts, φ_1 satisfying Laplace's equation except in body 1 and φ_2 except in body 2, then condition (5) for body 1 requires that

$$\int\int\left[\frac{\partial\varphi_1}{\partial\bar z}\nabla^2\varphi_1+\frac{\partial\varphi_2}{\partial\bar z}\nabla^2\varphi_1\right]\bar r d\bar r d\bar z=0.$$

The first term vanishes by (i); the second cannot vanish by virtue of (ii). The theorem shows that there is no static solution in general relativity for two bodies, each containing matter of one sign, situated on opposite sides of a surface $z=\text{const}$, with the metric tending to the Minkowski metric at infinity. It will now be shown that if the last condition is dropped such a solution is no longer impossible.

3.

The uniformly accelerated metric

$$ds^2=z^2dt^2-dx^2-dy^2-dz^2 \quad (10)$$

may be transformed to the Weyl-Levi-Civita form (1) by the transformation

$$\left.\begin{aligned} t&=\bar t\\ z&=e^{\varphi}\\ (x^2+y^2)^{\frac12}&=\bar r e^{-\varphi}\\ \tan^{-1}y/x&=\theta\end{aligned}\right\} \quad (11)$$

where

$$\varphi=\tfrac12\log\{[\bar r^2+(\bar z-a)^2]^{\frac12}+(\bar z-a)\} \quad (12)$$

$$\sigma=\tfrac12\log\{\tfrac12+\tfrac12(\bar z-a)[\bar r^2+(\bar z-a)^2]^{-\frac12}\} \quad (13)$$

and a is an arbitrary constant.

This φ, which will be called φ_0, satisfies Laplace's equation in the whole relevant portion of space, i.e., except on $\bar{r}=0$, $\bar{z}\leqslant a$. As was discussed earlier, the uniformly accelerated metric only represents part of space-time and so the singularity at $\bar{r}=0$, $\bar{z}\leqslant a$ need not cause surprise or alarm.

Note that, for $\bar{z}>a$, $\partial\varphi_0/\partial\bar{z}>0$.

Consider now the problem of two bodies, as in (iii) above, but add the potential φ_0. This implies that we drop the boundary condition that space is Minkowskian at infinity and replace it by that appropriate to a uniformly accelerated frame of reference in a space-time which is flat at infinity. If each of the bodies is entirely in $\bar{z}>a$, and if in the body (body 1) in the lower $\bar{z}$ region $\nabla^2\varphi_1\leqslant 0$, whereas in the other one (body 2) $\nabla^2\varphi_2\geqslant 0$, then condition (5) can be satisfied in suitable circumstances. For then, inside body 1, $\partial\varphi_2/\partial\bar{z}<0$ and, in body 2, $\partial\varphi_1/\partial\bar{z}<0$. As $\partial\varphi_0/\partial\bar{z}>0$ in both regions, there is now no argument from signs to show that the arrangement is impossible.

To establish the possibility we proceed as follows. Let B_1, B_2 be two finite regions of space with B_1 entirely in $\bar{z}_1\leqslant\bar{z}\leqslant\bar{z}_1'$ and B_2 in $\bar{z}_2\leqslant\bar{z}\leqslant\bar{z}_2'$ where $\bar{z}_1'<\bar{z}_2$. Let φ_1 satisfy Laplace's equation everwhere outside B_1, with $\varphi_1\rightarrow 0$ at infinity and with $\nabla^2\varphi_1\leqslant 0$ in B_1. Similarly let φ_2 satisfy Laplace's equation outside B_2, let it tend to zero at infinity, but $\nabla^2\varphi_2\geqslant 0$ in B_2. Then consider

$$\varphi=\varphi_0+k\varphi_1+l\varphi_2 \tag{14}$$

where the a entering φ_0 is a constant and k, l are constants to be determined later, with l, $k>0$ and $a<\bar{z}_1$. If condition (5) is to be satisfied for both B_1 and B_2 then

$$k\iint_{B_1}\bar{r}d\bar{r}d\bar{z}\nabla^2\varphi_1\left[\frac{\partial\varphi_0}{\partial\bar{z}}+l\frac{\partial\varphi_2}{\partial\bar{z}}\right]=0 \qquad (15)\dagger$$

and

$$l\iint_{B_2}\bar{r}d\bar{r}d\bar{z}\nabla^2\varphi_2\left[\frac{\partial\varphi_0}{\partial\bar{z}}+k\frac{\partial\varphi_1}{\partial\bar{z}}\right]=0. \qquad (16)\dagger$$

Since the factor k can be canceled in (15), this equation can be considered as an equation determining l. Moreover, since in B_1

$$\frac{\partial\varphi_0}{\partial\bar{z}}>0>\frac{\partial\varphi_2}{\partial\bar{z}}, \tag{17}$$

l will be positive. Similarly (16) determines a positive k.

All the conditions of the problem, therefore, are satisfied and so we have succeeded in constructing a uniformly accelerated pair of bodies whose densities have opposite sign.

† The Newtonian "self-force" term $\nabla^2\varphi_s\partial\varphi_s/\partial\bar{z}$ has been omitted in (15) and (16), since its integral vanishes by theorem (i).

As an example we may take $a=0$ and

$$\varphi_1=\begin{cases}-\dfrac{m_1}{[\bar{r}^2+(\bar{z}-h_1)^2]^{\frac{1}{2}}} & (\bar{r}^2+(\bar{z}-h_1)^2\geqslant a_1^2)\\[2ex] \dfrac{m_1}{2a_1^3}[\bar{r}^2+(\bar{z}-h_1)^2]-\dfrac{3m_1}{2a_1} & (\bar{r}^2+(\bar{z}-h_1)^2\leqslant a_1^2)\end{cases} \tag{18}$$

$$\varphi_2=\begin{cases}-\dfrac{m_2}{[\bar{r}^2+(\bar{z}-h_2)^2]^{\frac{1}{2}}} & (\bar{r}^2+(\bar{z}-h_2)^2\geqslant a_2^2)\\[2ex] \dfrac{m_2}{2a_2^3}[\bar{r}^2+(\bar{z}-h_2)^2]-\dfrac{3m_2}{2a_2} & (\bar{r}^2+(\bar{z}-h_2)^2\leqslant a_2^2)\end{cases}. \tag{19}$$

Here, for simplicity, h_1 and h_2 are disposable rather than k and l.

Condition (16) becomes, since $\nabla^2\varphi_2=\text{const}$ in B_2,

$$0=\iint_{B_2}\bar{r}d\bar{r}d\bar{z}\left[\frac{\partial\varphi_0}{\partial\bar{z}}+\frac{\partial\varphi_1}{\partial\bar{z}}\right]. \tag{20}$$

If $a_2\ll h_2-h_1$ then this becomes effectively

$$\left[\frac{\partial\varphi_0}{\partial\bar{z}}+\frac{\partial\varphi_1}{\partial\bar{z}}\right]_{\bar{z}=h_2,\bar{r}=0}=0 \tag{21}$$

and so

$$\frac{1}{2h_2}=-\frac{m_1}{(h_2-h_1)^2}. \tag{22}$$

Hence m_1 is negative.

Similarly we must have, if $a_1\ll h_2-h_1$ and $a_1\ll h_1$,

$$\left(\frac{\partial\varphi_0}{\partial\bar{z}}+\frac{\partial\varphi_2}{\partial\bar{z}}\right)_{\bar{z}=h_1,\bar{r}=0}=0 \tag{23}$$

$$\frac{1}{2h_1}=\frac{m_2}{(h_2-h_1)^2}. \tag{24}$$

It follows that in both bodies the derivatives of φ_0 are of the same order of magnitude as the derivatives of the potentials produced by themselves. If, then, the Newtonian potentials of both bodies are small, the derivatives of σ will be small of the second order, in spite of the additional terms due to φ_0. Accordingly the densities are an order of magnitude larger than the stresses, and are themselves, to the first order, given by $\kappa\rho=2e^{2(\varphi_0-\sigma_0)}\nabla^2\varphi$.

By (12) and (13)

$$e^{-2(\varphi_0-\sigma_0)}=\frac{1}{2(\bar{r}^2+\bar{z}^2)^{\frac{1}{2}}}\doteqdot\frac{1}{2h}. \tag{25}$$

Hence

$$\kappa\rho=12h\frac{m}{a^3}. \tag{26}$$

It may be advantageous to view the system from the Galilean frame of reference τ, ζ, α, θ at $\tau=\bar{t}=0$.‡ Then the bodies appear to be spheres with centers at $\zeta=(2h)^{\frac{1}{2}}$ and of radii $a(2h)^{-\frac{1}{2}}$. The densities are still given by (26) and so the masses are $M=m(2h)^{-\frac{1}{2}}$, while the accelerations are $(2h)^{-\frac{1}{2}}$. In the Newtonian limit, the accelerations should be given by

$$\frac{1}{(2h_2)^{\frac{1}{2}}}=f_2=-\frac{M_1}{[(2h_2)^{\frac{1}{2}}-(2h_1)^{\frac{1}{2}}]^2}=-\frac{m_1}{(2h_1)^{\frac{1}{2}}[(2h_2)^{\frac{1}{2}}-(2h_1)^{\frac{1}{2}}]^2}. \tag{27}$$

By (22) this equals

$$\frac{(h_2-h_1)^2}{2h_2(2h_1)^{\frac{1}{2}}[(2h_2)^{\frac{1}{2}}-(2h_1)^{\frac{1}{2}}]^2}=\frac{1}{(2h_2)^{\frac{1}{2}}}\frac{\{\frac{1}{2}[(2h_1)^{\frac{1}{2}}+(2h_2)^{\frac{1}{2}}]\}^2}{(2h_1)^{\frac{1}{2}}(2h_2)^{\frac{1}{2}}}. \tag{28}$$

This will be approximately the case provided $h_2-h_1\ll h_1$; i.e., provided the product of acceleration and distance apart is small. This is an appropriate limitation for the Newtonian case. It is interesting to note that the masses are not quite equal and opposite, but this is not surprising since their accelerations must be unequal in the uniformly accelerated model.

4.

The metric constructed in the preceding section contains a singularity at $\bar{r}=0$, $\bar{z}\leqslant 0$. This is not surprising, since the uniformly accelerated metric $\varphi=\varphi_0$ also contains such a singularity which is, however, purely artificial and is transformed away by returning to the $(\tau,\zeta,\alpha,\theta)$ metric. Is a similar elimination of the singularity possible if $\varphi=\varphi_0+\varphi_1+\varphi_2$?

We note first of all that if $\varphi=\varphi_0$ the entire $(\bar{t},\bar{z},\bar{r},\theta)$ space corresponds to $z\geqslant 0$ in (t,z,r,θ) and to a quarter of space-time; *viz.*, $\zeta\geqslant|\tau|$ in $(\tau,\zeta,\alpha,\theta)$. Keeping to the same transformation equations

$$\bar{r}=rz,\quad \bar{z}=\tfrac{1}{2}(z^2-r^2),\quad \bar{t}=t \tag{29}$$

we have now the metric

$$ds^2=z^2e^{2\psi}dt^2-e^{2(\delta-\psi)}(dr^2+dr^2)-r^2e^{-2\psi}d\theta^2 \tag{30}$$

where the singular part of the metric appears explicitly and we are only dealing with $z\geqslant 0$. It is easily seen that

$$\psi(z,r)=\varphi(\bar{z},\bar{r})-\varphi_0(\bar{z},\bar{r}) \tag{31}$$

is a regular function of z and r. Moreover it is readily established that, for small z, ψ can be expanded as a power series in z^2 with coefficients depending on r, the coefficients themselves being representable by power series in r^2 for small r. Also

$$\delta(z,r)=\sigma(\bar{z},\bar{r})-\sigma_0(\bar{z},\bar{r}) \tag{32}$$

is of the same nature.

Defining now

$$\tau=ze^{\psi}\sinh t,\quad \zeta=ze^{\psi}\cosh t,\quad \alpha=re^{-\psi} \tag{33}$$

we arrive at the metric

$$ds^2=d\tau^2-d\zeta^2-\mu(\tau d\tau-\zeta d\zeta)^2-2\nu d\alpha(\tau d\tau-\zeta d\zeta)-\lambda d\alpha^2-\alpha^2d\theta^2 \tag{34}$$

where the coefficients μ, ν, λ are functions only of α and $\zeta^2-\tau^2$. Note that this metric is invariant under any (ζ,τ) Lorentz transformation, showing that we are still dealing with a case of uniform acceleration though we are no longer in flat space-time.

By virtue of Eqs. (33), metric (34) is established only for $\zeta\geqslant|\tau|$. What happens on and beyond this boundary? It is clear from the structure of (34) that the boundary consists of parts of two null geodesics. Furthermore, a somewhat laborious comparison of coefficients yields expressions for μ, ν, λ in terms of ψ, $\partial\psi/\partial z$, $\partial\psi/\partial r$, and δ. An examination of these expressions establishes that, as a consequence of the behavior of ψ and δ for small z referred to above, the three new coefficients can be expanded in series of powers of $\zeta^2-\tau^2$ near the boundary, the coefficients of the terms in these series being functions of α that themselves can be expanded in powers of α^2 near $\alpha=0$. Metric (34) is therefore perfectly regular at the boundary.

As the boundary is a null geodesic the continuation of the metric beyond it is not uniquely defined. It would be most attractive to find a continuation of (34) that was free of singularities and of matter, and comprised all space-time. The mathematical difficulties of finding such a continuation appear to be formidable and have so far proved unsurmountable. It is also an interesting and significant problem to establish the existence of such a continuation, but this too has so far defied solution.

A solution of a different character has however been obtained. If (34) is supposed to retain its form for all τ and ζ then a metric is obtained that is symmetrical about $\tau=0$ and about $\zeta=0$. In the region $\tau\geqslant|\zeta|$ the coefficients depend only on α and on the *time like* variable $\tau^2-\zeta^2$ (and similarly in $|\zeta|\leqslant-\tau$) while in $-\zeta\geqslant|\tau|$ the mirror image of the conditions in $\zeta\geqslant|\tau|$ will apply; i.e., there will again be two uniformly accelerated bodies of opposite sign of mass. The sole question to be decided in order to establish the validity of this type of solution is whether there is an empty space metric of this kind in $\tau\geqslant|\zeta|$ fitting with the required degree of smoothness on to our previously obtained metric at $\tau=\zeta\geqslant 0$.

We first notice that the transformation

$$\tau=Te^{\epsilon}\cosh Z,\quad \zeta=Te^{\epsilon}\sinh Z,\quad \alpha=Re^{-\epsilon}$$

‡ Where $\xi=\alpha\cos\theta$, $\eta=\alpha\sin\theta$.

with

$$\epsilon=\epsilon(T,R) \tag{35}$$

yields the metric

$$ds^2=e^{2(\eta-\epsilon)}(dT^2-dR^2)-T^2e^{2\epsilon}dZ^2-R^2e^{-2\epsilon}d\theta^2 \tag{36}$$

with

$$\eta=\eta(T,R)$$

provided the μ, ν, λ of (34) are suitably connected with the ϵ, η of (36). In fact it turns out that the expressions for μ, ν, λ in terms of ϵ and η are identical with those in terms of ψ and δ provided ψ, δ, r, z^2, $z\partial/\partial z$ are replaced respectively by ϵ, η, R, $-T^2$, $T\partial/\partial T$. Accordingly, we have found a metric fitting smoothly to our original metric by continuing the intermediate metric (34) across the boundary.

The behavior of (36) away from the boundary is governed in empty space by the equations

$$\frac{\partial^2\epsilon}{\partial R^2}+\frac{1}{R}\frac{\partial\epsilon}{\partial R}=\frac{\partial^2\epsilon}{\partial T^2}+\frac{1}{T}\frac{\partial\epsilon}{\partial T} \tag{37}$$

$$(R^2-T^2)\frac{\partial\eta}{\partial T}=R^2T\left[\left(\frac{\partial\epsilon}{\partial R}\right)^2+\left(\frac{\partial\epsilon}{\partial T}\right)^2\right]-2RT^2\frac{\partial\epsilon}{\partial R}\frac{\partial\epsilon}{\partial T}+2R^2\frac{\partial\epsilon}{\partial T}-2RT\frac{\partial\epsilon}{\partial R} \tag{38}$$

$$-(R^2-T^2)\frac{\partial\eta}{\partial R}=RT^2\left[\left(\frac{\partial\epsilon}{\partial R}\right)^2+\left(\frac{\partial\epsilon}{\partial T}\right)^2\right]-2R^2T\frac{\partial\epsilon}{\partial R}\frac{\partial\epsilon}{\partial T}+2RT\frac{\partial\epsilon}{\partial T}-2R^2\frac{\partial\epsilon}{\partial R}. \tag{39}$$

Since (37) is a hyperbolic equation it will have no singularities for $T>0$, $R\geqslant 0$ provided ϵ and $\partial\epsilon/\partial T$ are given on $T=0$ in a nonsingular manner as even functions of R. This is the case in view of the properties of ψ and since ϵ is connected with ψ as mentioned above. In fact, near $T=0$, ϵ can be expanded in a power series in T^2 with coefficients that themselves can be expanded in powers of R^2, as a consequence of the corresponding property of ψ. Accordingly ϵ will be nonsingular. As for η, the only doubt about its nonsingular character arises from the factor R^2-T^2. It is, however, easily seen that this factor does not lead to a singularity provided $\partial\epsilon/\partial R=\partial\epsilon/\partial T$ on $R=T$, and this in turn is a consequence of (37) and of the fact that $\partial\epsilon/\partial R=\partial\epsilon/\partial T=0$ at $R=T=0$.

We have succeeded, therefore, in constructing a world-wide nonsingular solution of Einstein's equations containing two oppositely accelerated pairs of bodies,§ each pair consisting of two bodies of opposite sign of mass. Since $T_0{}^0$ and m are for any one body of the same sign the negative mass occurring is of type (iv).

§ This solution is closely analogous to Born's solution for the electromagnetic case [H. Bondi and T. Gold, Proc. Roy. Soc. (London) **A229**, 416 (1955)].