respect the symmetry of space inversion, so these particles are given different names: *neutrinos* for helicity $+1/2$, and *antineutrinos* for helicity $-1/2$.

Even though the helicity of a massless particle is Lorentz-invariant, the state itself is not. In particular, because of the helicity-dependent phase factor $\exp(i\sigma\theta)$ in Eq. (2.5.42), a state formed as a linear superposition of one-particle states with opposite helicities will be changed by a Lorentz transformation into a different superposition. For instance, a general one-photon state of four-momenta may be written

$$\Psi_{p;\alpha} = \alpha_+ \Psi_{p,+1} \; + \; \alpha_- \Psi_{p,-1} \; ,$$

where

$$|\alpha_+|^2 \; + \; |\alpha_-|^2 = 1 \; .$$

The generic case is one of *elliptic polarization*, with $|\alpha_\pm|$ both non-zero and unequal. *Circular polarization* is the limiting case where either α_+ or α_- vanishes, and *linear polarization* is the opposite extreme, with $|\alpha_+| = |\alpha_-|$. The overall phase of α_+ and α_- has no physical significance, and for linear polarization may be adjusted so that $\alpha_- = \alpha_+^*$, but the relative phase is still important. Indeed, for linear polarizations with $\alpha_- = \alpha_+^*$, the phase of α_+ may be identified as the angle between the plane of polarization and some fixed reference direction perpendicular to $\mathbf{p}$. Eq. (2.5.42) shows that under a Lorentz transformation $\Lambda^\mu{}_\nu$, this angle rotates by an amount $\theta(\Lambda, p)$. Plane polarized gravitons can be defined in a similar way, and here Eq. (2.5.42) has the consequence that a Lorentz transformation Λ rotates the plane of polarization by an angle $2\theta(\Lambda, p)$.

2.6 Space Inversion and Time-Reversal

We saw in Section 2.3 that any homogeneous Lorentz transformation is either proper and orthochronous (i.e., $\text{Det}\Lambda = +1$ and $\Lambda^0{}_0 \geq +1$) or else equal to a proper orthochronous transformation times either $\mathscr{P}$ or $\mathscr{T}$ or $\mathscr{P}\mathscr{T}$, where $\mathscr{P}$ and $\mathscr{T}$ are the space inversion and time-reversal transformations

$$\mathscr{P}^\mu{}_\nu = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} , \quad \mathscr{T}^\mu{}_\nu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} .$$

It used to be thought self-evident that the fundamental multiplication rule of the Poincaré group

$$U(\bar{\Lambda}, \bar{a}) \; U(\Lambda, a) = U(\bar{\Lambda}\Lambda, \bar{\Lambda}a + \bar{a})$$

would be valid even if Λ and/or $\bar{\Lambda}$ involved factors of $\mathscr{P}$ or $\mathscr{T}$ or $\mathscr{P}\mathscr{T}$. In particular, it was believed that there are operators corresponding to $\mathscr{P}$ and $\mathscr{T}$ themselves:

$$\mathsf{P} \equiv U(\mathscr{P}, 0) \qquad \mathsf{T} \equiv U(\mathscr{T}, 0)$$

such that

$$\mathsf{P}U(\Lambda, a)\mathsf{P}^{-1} = U(\mathscr{P}\Lambda\mathscr{P}^{-1}, \mathscr{P}a)\,, \tag{2.6.1}$$

$$\mathsf{T}\,U(\Lambda, a)\mathsf{T}^{-1} = U(\mathscr{T}\Lambda\mathscr{T}^{-1}, \mathscr{T}a) \tag{2.6.2}$$

for any proper orthochronous Lorentz transformation $\Lambda^\mu{}_\nu$ and translation a^μ. These transformation rules incorporate most of what is meant when we say that P or T are 'conserved'.

In 1956–57 it became understood[8] that this is true for P only in the approximation in which one ignores the effects of weak interactions, such as those that produce nuclear beta decay. Time-reversal survived for a while, but in 1964 there appeared indirect evidence[9] that these properties of T are also only approximately satisfied. (See Section 3.3.) In what follows, we will make believe that operators P and T satisfying Eqs. (2.6.1) and (2.6.2) actually exist, but it should be kept in mind that this is only an approximation.

Let us apply Eqs. (2.6.1) and (2.6.2) in the case of an infinitesimal transformation, i.e.,

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu \qquad a^\mu = \epsilon^\mu$$

with $\omega_{\mu\nu} = -\omega_{\nu\mu}$ and ϵ_μ both infinitesimal. Using (2.4.3), and equating coefficients of $\omega_{\rho\sigma}$ and ϵ_ρ in Eqs. (2.6.1) and (2.6.2), we obtain the P and T transformation properties of the Poincaré generators

$$\mathsf{P}\,iJ^{\rho\sigma}\mathsf{P}^{-1} = i\mathscr{P}_\mu{}^\rho\mathscr{P}_\nu{}^\sigma J^{\mu\nu}\,, \tag{2.6.3}$$

$$\mathsf{P}\,iP^\rho\mathsf{P}^{-1} = i\mathscr{P}_\mu{}^\rho P^\mu\,, \tag{2.6.4}$$

$$\mathsf{T}\,iJ^{\rho\sigma}\mathsf{T}^{-1} = i\mathscr{T}_\mu{}^\rho\mathscr{T}_\nu{}^\sigma J^{\mu\nu}\,, \tag{2.6.5}$$

$$\mathsf{T}\,iP^\rho\mathsf{T}^{-1} = i\mathscr{T}_\mu{}^\rho P^\mu\,. \tag{2.6.6}$$

This is much like Eqs. (2.4.8) and (2.4.9), except that we have not cancelled factors of i on both sides of these equations, because at this point we have not yet decided whether P and T are linear and unitary or antilinear and antiunitary.

The decision is an easy one. Setting $\rho = 0$ in Eq. (2.6.4) gives

$$\mathsf{P}\,iH\mathsf{P}^{-1} = iH\,,$$

where $H \equiv P^0$ is the energy operator. If P were antiunitary and antilinear then it would anticommute with i, so $\mathsf{P}H\mathsf{P}^{-1} = -H$. But then for any state Ψ of energy $E > 0$, there would have to be another state $\mathsf{P}^{-1}\Psi$ of energy $-E < 0$. There are no states of negative energy (energy less than that of the vacuum), so we are forced to choose the other alternative: P *is linear and unitary, and commutes rather than anticommutes with H.*

On the other hand, setting $\rho = 0$ in Eq. (2.6.6) yields

$$\mathsf{T}\, iH\mathsf{T}^{-1} = -iH\,.$$

If we supposed that T is linear and unitary then we could simply cancel the is, and find $\mathsf{T}H\mathsf{T}^{-1} = -H$, with the again disastrous conclusion that for any state Ψ of energy E there is another state $\mathsf{T}^{-1}\Psi$ of energy $-E$. To avoid this, we are forced here to conclude that T *is antilinear and antiunitary.*

Now that we have decided that P is linear and T is antilinear, we can conveniently rewrite Eqs. (2.6.3)–(2.6.6) in terms of the generators (2.4.15)–(2.4.17) in a three-dimensional notation

$$\mathsf{P}\mathbf{J}\mathsf{P}^{-1} = +\mathbf{J}\,, \tag{2.6.7}$$

$$\mathsf{P}\mathbf{K}\mathsf{P}^{-1} = -\mathbf{K}\,, \tag{2.6.8}$$

$$\mathsf{P}\mathbf{P}\mathsf{P}^{-1} = -\mathbf{P}\,, \tag{2.6.9}$$

$$\mathsf{T}\mathbf{J}\mathsf{T}^{-1} = -\mathbf{J}\,, \tag{2.6.10}$$

$$\mathsf{T}\mathbf{K}\mathsf{T}^{-1} = +\mathbf{K}\,, \tag{2.6.11}$$

$$\mathsf{T}\mathbf{P}\mathsf{T}^{-1} = -\mathbf{P}\,, \tag{2.6.12}$$

and, as shown before,

$$\mathsf{P}H\mathsf{P}^{-1} = \mathsf{T}H\mathsf{T}^{-1} = H\,. \tag{2.6.13}$$

It is physically sensible that P should preserve the sign of $\mathbf{J}$, because at least the orbital part is a vector product $\mathbf{r} \times \mathbf{p}$ of two vectors, both of which change sign under an inversion of the spatial coordinate system. On the other hand, T reverses $\mathbf{J}$, because after time-reversal an observer will see all bodies spinning in the opposite direction. Note by the way that Eq. (2.6.10) is consistent with the angular-momentum commutation relations $\mathbf{J} \times \mathbf{J} = i\mathbf{J}$, because T reverses not only $\mathbf{J}$, but also i. The reader can easily check that Eqs. (2.6.7)–(2.6.13) are consistent with all the commutation relations (2.4.18)–(2.4.24).

Let us now consider what P and T do to one-particle states:

$\mathsf{P}: M > 0$

The one-particle states $\Psi_{k,\sigma}$ are defined as eigenvectors of $\mathbf{P}, H$, and J_3 with eigenvalues $0, M$, and σ, respectively. From Eqs. (2.6.7), (2.6.9), and (2.6.13), we see that the same must be true of the state $\mathsf{P}\Psi_{k,\sigma}$, and therefore (barring degeneracies) these states can only differ by a phase

$$\mathsf{P}\Psi_{k,\sigma} = \eta_\sigma \Psi_{k,\sigma}$$

with a phase factor ($|\eta| = 1$) that may or may not depend on the spin σ.